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About some symmetric inequalities in power-exponential domain

Arkady M. Alt [9]

ABSTRACT. Here we will consider several problems related to inequalities of the form $H(x,y) \lesseqgtr e$, where $H(x,y)$ is symmetric function and x, y are real unequal positive numbers such that $x^y = y^x$.

MAIN RESULTS

i. Parametric representation of non trivial (unequal)positive solution of equation

$$x^y = y^x.$$

Let

$$D := \{(x,y) \mid x, y \in \mathbb{R}_+,\ x \neq y \text{ and } x^y = y^x\}$$

(obvious that $x \neq 1$ and $y \neq 1$).
Since D is symmetrical then $D = D_< \cup D_>$, where

$$D_< := \{(x,y) \mid x, y \in D \text{ and } x < y\}$$

and

$$D_> := \{(x,y) \mid x, y \in D \text{ and } x > y\}.$$

Let $f(x) := x^{\frac{1}{x}}, x > 0$. Since

$$f'(x) = \frac{f(x)}{x^2}(1 - \ln x)$$

then $f(x)$ strictly increasing on $(0, e]$ and strictly decreasing on $[0, \infty)$ with

$$\max_{x>0} f(x) = f(e) = e^{\frac{1}{e}}.$$

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Therefore, noting that

$$x^y = y^x \iff x^{\frac{1}{x}} = y^{\frac{1}{y}} \iff f(x) = f(y),$$

we can conclude that

$$(x, y) \in D_< \implies x < e < y,$$

and $(x, y) \in D_> \implies y < e < x$. (otherwise if x, y both belong to $(0, e)$ or (e, ∞) then, due to monotonicity $f(x)$ equality $f(x) = f(y)$ yields $x = y$,that is the contradiction. And in case $x = e$,since $y \neq e$ we again obtain contradiction $f(e) = f(y) \neq f(e)$).
Also note that if $(x, y) \in D_<$ then $x > 1$.Indeed, since

$$x^y = y^x \iff y = x \log_x y,$$

then supposition $x < 1$ implies $0 < \dfrac{y}{x} = \log_x y < 0$, i.e. contradiction.
Hence $(x, y) \in D_< \implies x \in (1, e),\ \ y \in (e, \infty)$ and then

$$(x, y) \in D \implies x, y \in (1, e) \cup (e, \infty).$$

Let $t := \log_x y - 1$.Then

$$\log_x y = t + 1 \iff y = x^{t+1}$$

and

$$y = x \log_x y \iff y = x(t + 1).$$

Hence $y = x^{t+1}$, and, therefore,

$$x^{t+1} = x(t + 1) \iff x^t = t + 1 \iff$$

$$x = (t + 1)^{\frac{1}{t}} \implies y = (t + 1)^{1+\frac{1}{t}},$$

where $t > 0$ (since $1 < x < y$)
Thus

$$D_< = \left\{ \left((t + 1)^{\frac{1}{t}}, (t + 1)^{1+\frac{1}{t}} \right) \mid t \in (0, \infty) \right\}$$

is set of all non-trivial solution of equation $x^y = y^x$, satisfied $x < y$ and, due to symmetry of $H(x, y)$, we have $H(D) = H(D_<)$ ($H(D)$ is range of $H(x, y)$ for $(x, y) \in D$)

Since $t = \log_x y - 1$ then

$$(x, y) \in D_> \implies 1 < y < e < x \implies$$

$$1 < y < x \iff \log_x 1 - 1 < \log_x y - 1 < \log_x x - 1 \iff -1 < t < 0.$$

Therefore,

$$D_> = \left\{ \left((t+1)^{\frac{1}{t}}, (t+1)^{1+\frac{1}{t}} \right) \mid t \in (-1, 0) \right\}$$

and

$$D = \left\{ \left((t+1)^{\frac{1}{t}}, (t+1)^{1+\frac{1}{t}} \right) \mid t \in (-1, 0) \cup (0, \infty) \right\}.$$

Note that

$$\lim_{t \to 0} x = \lim_{t \to 0} (t+1)^{\frac{1}{t}} = e$$

and

$$\lim_{t \to 0} y = \lim_{t \to 0} (t+1)^{1+\frac{1}{t}} = e.$$

Since $x < e < y$ then x and y approaches to e from left and right respectively. We also have

$$\lim_{t \to \infty} x = \lim_{t \to \infty} (t+1)^{\frac{1}{t}} = 1$$

and

$$\lim_{t \to \infty} y = \lim_{t \to \infty} (t+1)^{1+\frac{1}{t}} = \infty.$$

Remark 1. Correspondence

$$t \mapsto x(t) = (t+1)^{\frac{1}{t}} : (0, \infty) \longrightarrow (1, e)$$

is one-to-one correspondence, moreover $x(t)$ is strictly decreasing on $(0, \infty)$. Indeed, since

$$\frac{t}{t+1} < \ln(1+t)$$

for any $t > 0$

$$\left(\frac{t}{t+1}-\ln(1+t)\right)' = \frac{1}{(t+1)^2} - \frac{1}{1+t} = \frac{-t}{(t+1)^2} < 0,$$

$t > 0$ implies

$$\frac{t}{t+1} - \ln(1+t) < \frac{0}{0+1} - \ln(1+0) = 0$$

then

$$(\ln x(t))' = \frac{1}{t(t+1)} - \frac{\ln(1+t)}{t^2} = \frac{1}{t^2}\left(\frac{t}{t+1} - \ln(1+t)\right) < 0, t > 0.$$

Thus for any $x \in (1,e)$ equation $x = (t+1)^{\frac{1}{t}}$ always have unique solution $t(x) \in (0,\infty)$ and then

$$y(x) := (t(x)+1)^{1+\frac{1}{t(x)}}$$

is function of x, such that

$$D_< = \{(x, y(x)) \mid x \in (1,e)\}$$

Remark 2. Parametrization of D which we obtain above isn't symmetric as would be expected given that D is symmetrical. But denoting

$$s := \frac{\ln y + \ln x}{\ln y - \ln x} = \frac{t+2}{t}$$

we obtain $t = \dfrac{2}{s-1}$ and then

$$\left[\begin{array}{c} 0 < t \\ -1 < t < 0 \end{array}\right. \iff \left[\begin{array}{c} s > 1 \\ -1 < \dfrac{2}{s-1} < 0 \end{array}\right. \iff \left[\begin{array}{c} s > 1 \\ s < -1 \end{array}\right.,$$

$$x = \left(\frac{s+1}{s-1}\right)^{\frac{s-1}{2}}, y = \left(\frac{s+1}{s-1}\right)^{\frac{s+1}{2}}.$$

Thus,

$$D = \left\{\left(\left(\frac{s+1}{s-1}\right)^{\frac{s-1}{2}}, \left(\frac{s+1}{s-1}\right)^{\frac{s+1}{2}}\right) \mid |s| > 1\right\}$$

and we also have

$$x(-s) = y(s), y(-s) = x(s)$$

for any $|s| > 1$.
Since

$$\lim_{s\to 1+} x(s) = \left(\frac{s+1}{s-1}\right)^{\frac{s-1}{2}} = \lim_{t\to\infty} x(t) = \lim_{t\to\infty} (t+1)^{\frac{1}{t}} = 1$$

and

$$\lim_{s\to 1+} y(s) = \lim_{s\to 1+} \left(\frac{s+1}{s-1}\right)^{\frac{s+1}{2}} = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} (t+1)^{\frac{1}{t}+1} = \infty$$

then $x = 1$ is vertical asymptote for $y(x)$;
Since

$$\lim_{s\to(-1)-} x(s) = \lim_{s\to(-1)-} y(-s) = \lim_{s\to 1+} y(s) = \lim_{s\to 1+} \left(\frac{s+1}{s-1}\right)^{\frac{s+1}{2}} =$$

$$= \lim_{t\to\infty} (t+1)^{\frac{1}{t}+1} = \infty$$

and

$$\lim_{s\to -1} y(s) = \lim_{s\to 1+} y(-s) = \lim_{s\to 1+} x(s) = \lim_{s\to 1+} \left(\frac{s+1}{s-1}\right)^{\frac{s+1}{2}} =$$

$$= \lim_{t\to\infty} x(t) = \lim_{t\to\infty} (t+1)^{\frac{1}{t}} = 1$$

then $y = 1$ is horizontal asymptote for $y(x)$.

ii. Problems. Problem 1.

a). Find $\inf_{(x,y)\in D} \sqrt{xy}$.

b). Find $\inf_{(x,y)\in D} \dfrac{x+y}{2}$.

c). Find $\inf\limits_{(x,y)\in D} (x+1)(y+1)$.

Problem 2. Find $\sup\limits_{(x,y)\in D} \left(\dfrac{x^{-1}+y^{-1}}{2}\right)^{-1}$.

Problem 3. Find $\inf\limits_{(x,y)\in D} \left(\dfrac{x^p+y^p}{2}\right)^{\frac{1}{p}}, p>0.$

Problem 4*. Find $\inf\limits_{(x,y)\in D} (x-1)(y-1)$.

Problem 5. Let $0<\alpha\le\beta$. Find $\inf\limits_{(x,y)\in D_<} x^\alpha y^\beta$.

Solutions.

Problem 1a. For $(x,y)\in D_<$ we have

$$xy = x^2(t+1) = (t+1)^{1+\frac{2}{t}}$$

then

$$(\ln xy)' = \left(\left(1+\frac{2}{t}\right)\ln(t+1)\right)' = \left(-\frac{2}{t^2}\right)\ln(t+1) + \left(1+\frac{2}{t}\right)\cdot\frac{1}{t+1} =$$

$$= \frac{1}{t^2}\left(\frac{t(t+2)}{t+1} - 2\ln(t+1)\right).$$

We will prove that

$$\frac{t(t+2)}{t+1} - 2\ln(t+1) > 0$$

for any $t>0$. Indeed,

$$\left(\frac{t(t+2)}{t+1} - 2\ln(t+1)\right)' = \left(t+1-\frac{1}{t+1} - 2\ln(t+1)\right)' =$$

$$= 1+\frac{1}{(t+1)^2} - \frac{2}{t+1} = \frac{t^2+2t+2-2t-2}{(t+1)^2} = \frac{t^2}{(t+1)^2} > 0.$$

Thus, $(\ln xy)' > 0$ and, therefore, $\ln xy$ is increasing on $(0,\infty)$.

Since xy is increasing on $(0,\infty)$ then

$$\inf_{(x,y)\in D}\sqrt{xy}=\inf_{(x,y)\in D_<}\sqrt{xy}=\lim_{t\to 0+}\sqrt{xy}=$$

$$=\lim_{t\to 0+}(t+1)^{\frac{1}{2}+\frac{1}{t}}=e.$$

Thus, $\sqrt{xy}>e$, where $x,y>0, x\neq y$ and $x^y=y^x$.

Remark. Sequence variant of correspondent inequality is $\left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}}>e, n\in\mathbb{N}$. For $t=\dfrac{1}{n}$ we have

$$\sqrt{xy}=\sqrt{x^2(t+1)}=x\sqrt{(t+1)}=\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}.$$

Of course inequality $\left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}}>e$ is immediate consequence from inequality $\sqrt{xy}>e$ but can be proved directly.

Since $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}}$ then suffice to prove that $\left(1+\dfrac{1}{n}\right)^{2n+1}$ is decreasing in $\mathbb{N}$ i.e. to prove inequality

$$\left(1+\frac{1}{n}\right)^{2n+1}>\left(1+\frac{1}{n+1}\right)^{2n+3}\iff$$

$$\iff\left(\frac{n+1}{n}\right)^{2n+1}>\left(\frac{n+2}{n+1}\right)^{2n+3}\iff$$

$$\iff\left(\frac{2n+2}{2n}\right)^{2n+1}>\left(\frac{2n+4}{2n+2}\right)^{2n+3}\iff$$

$$\iff\left(\frac{m+1}{m-1}\right)^{m}>\left(\frac{m+3}{m+1}\right)^{m+2},$$

where $m:=2n+1$.
Using Extended Bernoulli Inequality:

$$(1+x)^n\geq 1+nx+\frac{n(n-1)}{2}x^2,\ n\in\mathbb{N},x>0$$

we will prove that for any $m\geq 2$ holds inequality

$$\left(\frac{m+1}{m-1}\right)^m > \left(\frac{m+2}{m}\right)^{m+1} \iff$$

$$\iff \left(\frac{m+1}{m-1}\right)^m \cdot \left(\frac{m}{m+2}\right)^m > \frac{m+2}{m} \iff$$

$$\iff \left(\frac{m^2+m}{m^2+m-2}\right)^m > \frac{m+2}{m} \iff \left(1+\frac{2}{m^2+m-2}\right)^m > 1+\frac{2}{m}. \tag{1}$$

Since

$$\left(1+\frac{2}{m^2+m-2}\right)^m >$$

$$> 1+\frac{2m}{(m-1)(m+2)}+\frac{4}{(m-1)^2(m+2)^2}\cdot\frac{m(m-1)}{2} =$$

$$= 1+\frac{2m}{(m-1)(m+2)}+\frac{2m}{(m-1)(m+2)^2}$$

suffice to prove

$$\frac{2m}{(m-1)(m+2)}+\frac{2m}{(m-1)(m+2)^2} > \frac{2}{m} \iff$$

$$\iff \frac{1}{m+2}+\frac{1}{(m+2)^2} > \frac{m-1}{m^2} \iff$$

$$\frac{m+3}{(m+2)^2} > \frac{m-1}{m^2} \iff (m+3)\,m^2 > (m+2)^2(m-1).$$

We have

$$(m+3)\,m^2-(m+2)^2(m-1) =$$

$$= m^3+3m^2-m^3-4m^2-4m+m^2+4m+4 = 4.$$

Using (1) twice we obtain

$$\left(\frac{m+1}{m-1}\right)^m > \left(\frac{m+2}{m}\right)^{m+1} > \left(\frac{m+3}{m+1}\right)^{m+2}.$$

Thus,

$$\left(\frac{m+1}{m-1}\right)^m > \left(\frac{m+3}{m+1}\right)^{m+2}.$$

Problem 1b. Since $\frac{x+y}{2} \geq \sqrt{xy}$ and $\lim_{t\to 0} \frac{x+y}{2} = e$ then, due to previous problem,

$$\inf_{(x,y)\in D} \frac{x+y}{2} = e.$$

Thus, $\frac{x+y}{2} > e$, where $x, y > 0, x \neq y$ and $x^y = y^x$.

Remark. Sequence variant of correspondent inequality is

$$e < \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{2n}\right), n \in \mathbb{N}.$$

For $t = \frac{1}{n}$ we have

$$\frac{x+y}{2} = \frac{x\,(t+2)}{2} = (1+t)^{\frac{1}{t}} \left(1+\frac{t}{2}\right) =$$

$$= \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{2n}\right) > e.$$

Direct proof. Since

$$\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{2n}\right) > \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} \iff 1+\frac{1}{2n} \geq \left(1+\frac{1}{n}\right)^{\frac{1}{2}} \iff$$

$$1+\frac{1}{n}+\frac{1}{4n^2} > 1+\frac{1}{n}.$$

Problem 1c. Suffice note that

$$(x+1)\,(y+1) \geq (\sqrt{xy}+1)^2$$

and apply inequality $\sqrt{xy} > e$.

Problem 2. Let

$$H\,(x,y) := \left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1}.$$

For $(x,y) \in D_<$ we have

$$H(x,y) = \frac{2xy}{x+y} = \frac{2xy}{(t+2)x} = \frac{2y}{t+2} =$$

$$= \frac{2(t+1)^{1+\frac{1}{t}}}{t+2} = e^{h(t)},$$

where

$$h(t) := \ln \frac{2(t+1)^{1+\frac{1}{t}}}{t+2} = \ln \frac{2t+2}{t+2} + \frac{\ln(t+1)}{t}, t \in (0,\infty).$$

Then

$$\sup_{(x,y)\in D} H(x,y) = \sup_{(x,y)\in D_<} H(x,y) = \sup_{t\in(0,\infty)} e^{h(t)}$$

Note that

$$h'(t) = -\frac{1}{t^2}\ln(t+1) + \frac{2}{t(t+2)} = \frac{1}{t^2}\left(\frac{2t}{t+2} - \ln(t+1)\right) < 0.$$

Indeed, since

$$\left(\frac{2t}{t+2} - \ln(t+1)\right)' = \frac{4}{(t+2)^2} - \frac{1}{t+1} = -\frac{t^2}{(t+2)^2}$$

then

$$\frac{2t}{t+2} - \ln(t+1)$$

is decreasing on $(0,\infty)$ and, therefore,

$$\frac{2t}{t+2} - \ln(t+1) < \lim_{t\to 0}\left(\frac{2t}{t+2} - \ln(t+1)\right) = 0.$$

Hence, $h(t)$ is decreasing on $(0,\infty)$ and, therefore,

$$h(t) < \lim_{t\to 0} h(t) = \lim_{t\to 0}\left(\ln\frac{2t+2}{t+2} + \frac{\ln(t+1)}{t}\right) =$$

$$= \lim_{t\to 0}\ln\frac{2t+2}{t+2} + \lim_{t\to 0}\frac{\ln(t+1)}{t} = 0 + 1 = 1$$

yields $\sup\limits_{t>0} h(t) = 1$.

Thus, $\sup_{(x,y)\in D} H(x,y) = e$ or, in form of inequality

$\left(\dfrac{x^{-1}+y^{-1}}{2}\right)^{-1} < e$, where $x, y > 0, x \neq y$ and $x^y = y^x$.

Remark. Sequence variant of correspondent inequality is

$$\left(1+\frac{1}{n}\right)^{n+1}\left(1+\frac{1}{2n}\right)^{-1} < e.$$

For $t = \dfrac{1}{n}$ we have

$$H(x,y) = \frac{2\left(1+\frac{1}{n}\right)^{n+1}\cdot n}{2n+1} = \frac{2(n+1)^{n+1}}{n^n(2n+1)} < e \iff$$

$$\left(1+\frac{1}{n}\right)^{n+1}\left(1+\frac{1}{2n}\right)^{-1} < e.$$

Problem 3. Since $\left(\dfrac{x^p+y^p}{2}\right)^{\frac{1}{p}} \geq \sqrt{xy}$ and $\lim_{t\to 0}\left(\dfrac{x^p+y^p}{2}\right)^{\frac{1}{p}} = e$ then (see Problem 1) $\inf_{(x,y)\in D}\left(\dfrac{x^p+y^p}{2}\right)^{\frac{1}{p}} = e$, that is $\left(\dfrac{x^p+y^p}{2}\right)^{\frac{1}{p}} > e$.
Sequence variant is:

$$\left(\frac{\left(1+\frac{1}{n}\right)^{pn}+\left(1+\frac{1}{n}\right)^{p(n+1)}}{2}\right)^{\frac{1}{p}} = \left(1+\frac{1}{n}\right)^{n}\left(\frac{1+\left(1+\frac{1}{n}\right)^{p}}{2}\right)^{\frac{1}{p}} > e.$$

Problem 4*. For $(x,y)\in D_<$ we have

$$(x-1)(y-1) = (x-1)(x(t+1)-1) =$$

$$= x(x-1)(t+1) - x + 1,$$

where $x = (1+t)^{\frac{1}{t}}$. Let

$$H(t) := x(x-1)(t+1) - x + 1.$$

We have

$$H'(t) = (2x-1)\,x'(t+1) + x^2 - x - x' =$$

$$= x'((2x-1)(t+1)-1) + x^2 - x =$$

$$= x'(2x(t+1)-(t+2)) + x^2 - x =$$

$$= x(2x(t+1)-(t+2))(\ln x)' + x^2 - x =$$

$$= x\left((2x(t+1)-(t+2))\left(\frac{\ln(1+t)}{t}\right)' + x - 1\right) =$$

$$= x\left((2x(t+1)-(t+2))\left(\frac{1}{t(1+t)} - \frac{\ln(1+t)}{t^2}\right) + x - 1\right) > 0.$$

Indeed, since $\ln(1+t) < t$ and, by Bernoulli Inequality,

$$(1+t)^{1+\frac{1}{t}} > 1 + \left(1+\frac{1}{t}\right)t = 2+t,$$

then we have

$$\left(\frac{1}{t(1+t)} - \frac{\ln(1+t)}{t^2}\right) + (t+1)^{\frac{1}{t}} - 1 >$$

$$> \left(\frac{1}{t(1+t)} - \frac{t}{t^2}\right) + (1+t)^{\frac{1}{t}} - 1 = \left(\frac{1}{t(1+t)} - \frac{1}{t}\right) + (t+1)^{\frac{1}{t}} - 1 =$$

$$= -\frac{1}{1+t} + (1+t)^{\frac{1}{t}} - 1 = (1+t)^{\frac{1}{t}} - \frac{t+2}{1+t} = \frac{(1+t)^{1+\frac{1}{t}} - (t+2)}{1+t} > 0$$

and

$$2x(t+1) - (t+2) = 2(1+t)^{1+\frac{1}{t}} - (t+2) > 4 + 2t - t - 2 = t + 2 > 0.$$

Since, $H(t)$ increasing on $(0,\infty)$ then

$$\inf_{(x,y)\in D} (x-1)(y-1) =$$

$$= \inf_{(x,y)\in D_<} (x-1)(y-1) = \lim_{t\to 0} H(t) = e^2 - 1.$$

Thus, $(x-1)(y-1) > e^2 - 1$, where $x, y > 0, x \neq y$ and $x^y = y^x$.

Remark 1. For $t = \dfrac{1}{n}$ we have

$$(x-1)(y-1) = \left(\left(1+\frac{1}{n}\right)^n - 1\right)\left(\left(1+\frac{1}{n}\right)^{n+1} - 1\right)$$

then sequence variant of correspondent inequality is

$$\left(\left(1+\frac{1}{n}\right)^n - 1\right)\left(\left(1+\frac{1}{n}\right)^{n+1} - 1\right) > (e-1)^2.$$

Since

$$\lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n - 1\right)\left(\left(1+\frac{1}{n}\right)^{n+1} - 1\right) = (e-1)^2$$

and

$$\left(\left(1+\frac{1}{n}\right)^n - 1\right)\left(\left(1+\frac{1}{n}\right)^{n+1} - 1\right) \downarrow \mathbb{N},$$

then

$$\left(\left(1+\frac{1}{n}\right)^n - 1\right)\left(\left(1+\frac{1}{n}\right)^{n+1} - 1\right) > (e-1)^2$$

or

$$\left(1+\frac{1}{n}\right)^n \left(\left(1+\frac{1}{n}\right)^{n+1} - 2\left(1+\frac{1}{2n}\right)\right) > e^2 - 2e.$$

Remark 2. Since

$$\lim_{t\to 0}\left(\frac{1}{t(1+t)} - \frac{\ln(1+t)}{t^2}\right) = \lim_{t\to 0}\left(\frac{t-(1+t)\ln(1+t)}{t^2(1+t)}\right) =$$

$$= \lim_{t\to 0}\left(\frac{t-(1+t)\left(t-\frac{t^2}{2}+o\left(t^2\right)\right)}{t^2}\right) =$$

$$= \lim_{t\to 0}\left(\frac{t-t+\frac{t^2}{2}-t^2+\frac{t^3}{2}}{t^2}\right) = -\frac{1}{2}$$

then

$$\lim_{t\to 0} H'(t) =$$

$$= \lim_{t\to 0} x\left(\left(2x(t+1)-(t+2)\right)\left(\frac{1}{t(1+t)}-\frac{\ln(1+t)}{t^2}\right)+x-1\right) =$$

$$= e\left((2e-2)\cdot\left(-\frac{1}{2}\right)+e-1\right) = 0.$$

Remark 3. Let $x \in (1,e)$ then obtained inequality can be rewritten as

$$(x-1)(y(x)-1) > (e-1)^2 \iff y(x) > \frac{(e-1)^2}{x-1}+1.$$

Let $g(x) := \dfrac{(e-1)^2}{x-1}+1$. Since

$$g(x) = \frac{(e-1)^2}{x-1}+1 > \frac{(e-1)^2}{e-1}+1 = e$$

then $y(x) > g(x) > e$ yields

$$y(x)^{\frac{1}{y(x)}} < g(x)^{\frac{1}{g(x)}}.$$

From the other hand we have

$$y(x)^x = x^{y(x)} \iff y(x)^{\frac{1}{y(x)}} = x^{\frac{1}{x}}.$$

Hence

$$x^{\frac{1}{x}} < g(x)^{\frac{1}{g(x)}} \iff x^{g(x)} < g(x)^x \iff$$

$$\iff x^{\frac{(e-1)^2}{x-1}+1} < \left(\frac{(e-1)^2}{x-1}+1\right)^x \iff$$

$$\iff \left(\frac{(e-1)^2}{x-1}+1\right)\ln x < x\left(\frac{(e-1)^2}{x-1}+1\right).$$

Problem 5. Since for $(x,y)\in D_<$ we have $x=(1+t)^{\frac{1}{t}}$ and $y=(1+t)^{1+\frac{1}{t}}$ then

$$x^\alpha y^\beta = (1+t)^{\frac{\alpha}{t}}(1+t)^{\beta+\frac{\beta}{t}} = (1+t)^{\frac{\alpha+\beta}{t}+\beta} = \left((1+t)^{\frac{1}{t}+\frac{\beta}{\alpha+\beta}}\right)^{\alpha+\beta}.$$

Since $\alpha\le\beta$ then

$$\frac{\beta}{\alpha+\beta}\ge\frac{\beta}{\beta+\beta}=\frac{1}{2}$$

and, therefore,

$$x^\alpha y^\beta \ge \left((1+t)^{\frac{1}{t}+\frac{1}{2}}\right)^{\alpha+\beta} > e^{\alpha+\beta}$$

(because $(1+t)^{\frac{1}{t}+\frac{1}{2}}$ is increasing on $(0,\infty)$ (see solution to Problem 1) and $\lim\limits_{t\to 0}(1+t)^{\frac{1}{t}+\frac{1}{2}}=e$). Also we have

$$\lim_{t\to 0} x^\alpha y^\beta = \lim_{t\to 0}(1+t)^{\frac{\alpha+\beta}{t}+\beta} = e^{\alpha+\beta}.$$

Thus,

$$\inf_{(x,y)\in D_<} x^\alpha y^\beta = e^{\alpha+\beta}.$$

E-mail: arkady.alt@gmail.com